Ergodic Theory and Measured Group Theory

Lecture 14

Lemma. If $M_0, M_1, \ldots \subseteq \mathbb{N}$ of density 0, then $\exists M_\infty \subseteq \mathbb{N}$ of density 0 s.t. $M_n \setminus M_\infty$ is finite for all n, and $d(M_\infty) = 0$. Call M_∞ the diagonal union of (M_n).

Proof. We will use that finite unions of density 0 sets has upper density 0; in particular, we assume WLOG that the M_n are increasing by replacing M_n with $\bigcup_{i \leq n} M_i$.

Let $N_0 \in \mathbb{N}$ be s.t. $\forall n \geq N_0$ $\frac{|M_0 \cap I_n|}{|I_n|} < \frac{1}{2^0}$.

Put $M_\infty|_{[0,N_0]} := M_0|_{[0,N_0]}$.

Next, let $N_1 > N_0$ be s.t. $\forall n \geq N_1$ $\frac{|M_1 \cap I_n|}{|I_n|} < \frac{1}{2^1}$.

Put $M_\infty|_{(N_0,N_1]} := M_1|_{(N_0,N_1]}$.

$\vdots$

Let N_k be s.t. $\forall n \geq N_k$ $\frac{|M_k \cap I_n|}{|I_n|} < \frac{1}{2^k}$ and

put $M_\infty|_{(N_{k-1},N_k]} := M_k|_{(N_{k-1},N_k]}$.

Then M_∞ contains each $M_k \setminus [0, N_{k-1}]$.
Let $\varepsilon > 0$ let k be s.t. $2^{-k} < \varepsilon$. Then $\forall n \geq N_k$, we have that $\frac{|M_\infty \cap I_n|}{|I_n|} < 2^{-k}$. This is because if $N_k \leq n < N_\ell$ then $M_\infty \cap I_n \subseteq M_\ell \cap I_n$ and $\frac{|M_\ell \cap I_n|}{|I_n|} < 2^{-\ell} < 2^{-k}$. $\square$

Lemma. If $f : \mathbb{N} \to [0, \infty)$ is a bounded function, then
$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(i) = 0 \iff \exists \text{ a } 0 \text{ density set } M \subseteq \mathbb{N} \text{ s.t. } \lim_{\substack{n\to\infty \\ n \notin M}} f(n) = 0.$$

Proof. $\Rightarrow$. For each $\varepsilon > 0$, let $A_\varepsilon := \{n \in \mathbb{N} : f(n) \geq \varepsilon\}$.

Claim. A_ε is density 0.

Proof. Let $\delta > 0$.

Let N_δ be s.t. $\forall n \geq N_\delta$ $\frac{1}{n+1} \sum_{i=0}^{n} f(i) < \varepsilon \cdot \delta$.

Then
$$\frac{1}{n+1} \sum_{i=0}^{n} f(i) \geq \frac{1}{n+1} \sum_{i=0}^{n} f(i)\, \mathbb{1}_{A_\varepsilon}(i)$$
$$\geq \varepsilon \cdot \frac{|A_\varepsilon \cap I_n|}{|I_n|}.$$

Hence $\dfrac{|A_\varepsilon \cap I_n|}{|I_n|} \leq \dfrac{\varepsilon\delta}{\varepsilon} = \delta.$ $\boxtimes$

Let A_∞ be as in the above lemma applied to $A_{\frac{1}{n}}$, i.e. $d(A_\infty)=0$ and $A_{1/n} \setminus A_\infty$ is finite.

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N$ s.t. $n \notin A_\infty$, $n \notin A_{\frac{1}{k}}$ where k is s.t. $\frac{1}{k} < \varepsilon$. Hence $f(n) < \frac{1}{k} < \varepsilon$.

$\Leftarrow$. Let M be the bad set of density 0.

$$\frac{1}{n+1}\sum_{i=0}^{n} f(i) = \frac{1}{n+1}\sum_{i=0}^{n} f(i)\mathbb{1}_{M^c}(i) + \frac{1}{n+1}\sum_{i=0}^{n} f(i)\mathbb{1}_{M}(i).$$

$\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n \geq N$ and $n \notin M$, $f(i) < \varepsilon$

hence $$\frac{1}{n+1}\sum_{i=0}^{n} f(i)\mathbb{1}_{M^c}(i) = \frac{1}{n+1}\sum_{i=0}^{N} \bigcirc + \frac{1}{n+1}\sum_{i=N+1}^{n} f(i)\mathbb{1}_{M^c}(i)$$

$$\underset{n\to\infty}{\leq} 0 + \varepsilon,$$ so for large enough n, $\frac{1}{n+1}\sum_{i=0}^{n} f(i)\mathbb{1}_{M^c}(i) < 2\varepsilon.$

$$\frac{1}{n+1}\sum_{i=0}^{n} f(i)\mathbb{1}_{M}(i) \leq \|f\|_\infty \cdot \frac{|M \cap I_n|}{|I_n|} \to 0 \text{ as } n\to\infty.$$ $\square$

Cor. A pmp transformation T on (X,μ) is weakly mixing (i.e. $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0$) if and only if $\exists$ density 0 set $M \subseteq \mathbb{N}$ s.t. $\lim_{\substack{n\to\infty \\ n\notin M}} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$. Moreover, since (X,μ) is standard, we can take $M \subseteq \mathbb{N}$ that works for all Borel $A, B \subseteq X$.

Proof. The $\Leftrightarrow$ is just by the above lemma applied to $f(n) := |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)|$. To see independence of M from the sets A, B, let $\mathcal{A}$ be a ctbl generating algebra and $\forall A, B \in \mathcal{A}$, let $M_{A,B}$ be the bad set. Let M be the diagonal union of the $M_{A,B}$, $A, B \in \mathcal{A}$. $\square$

Theorem (More equivalences to weak mixing). For a pmp transformation T on a st. prob. space (X,μ), TFAE:

(1) T is weakly mixing.

(1') $\forall f, g \in L^2(X,\mu)$, $\frac{1}{n+1} \sum_{i=0}^{n} |\langle T^n f, g\rangle - \int f \int g| \to 0$ as $n \to \infty$.

(2) $\exists$ density 0 set $M \subseteq \mathbb{N}$ s.t. $\lim_{\substack{n\to\infty \\ n \notin M}} \mu(T^{-n}A \cap B) = \mu(A)\cdot\mu(B)$ $\forall$ Borel A, B.

(2') $\exists$ density 0 set $M \subseteq \mathbb{N}$ s.t. $\lim_{\substack{n\to\infty \\ n\notin M}} \langle T^n f, g\rangle = \int f \cdot \int g$.

(3) $T \times S$ on $(X \times Y, \mu \times \nu)$ is ergodic for any ergodic pmp S on (Y,ν).

(4) $T \times T$ on $(X \times X, \mu \times \mu)$ is ergodic.

(5) $T \times T$ on $(X \times X, \mu \times \mu)$ is weakly mixing.

(6) $\forall f \in L^2(X, \mu)$, if $\{T^n f : n \in \mathbb{N}\}$ is precompact in $L^2(X, \mu)$, then f is constant a.e.

Proof. (1) $\Leftrightarrow$ (3). Done.

(3) $\Leftrightarrow$ (3'). Almost done in homework.

(3') $\Leftrightarrow$ (2). Done.

(2) $\Leftarrow$ (1). Trivial.

(2) $\Rightarrow$ (5). To show ergodicity, it is enough to show that the von Neumann mean ergodic theorem holds for $T \times S$, and it is enough to check this for a generating algebra of functions in $L^2(X \times Y, \mu \times \nu)$. Such an algebra is that of functions of the form $(x,y) \mapsto f_1(x) f_2(y)$ for some $f_1 \in L^2(X, \mu)$ and $f_2 \in L^2(Y, \nu)$.

It is also enough to check this for functions with mean 0, i.e. let $g(x,y) := g_1(x) \cdot g_2(y)$ be s.t. $\int_{X \times Y} g \, d\mu \times \nu = 0$.

Since $\int\limits_{X\times Y} g \, d\mu\times\nu = \left(\int\limits_X g_1 \, d\mu\right)\cdot\left(\int\limits_Y g_2 \, d\nu\right)$, one of g_1 and g_2 is mean 0. Let $f(x,y) := f_1(x)\cdot f_2(y)$. We show that

$$\frac{1}{n+1}\sum_{i=0}^{n} \langle (T\times S)^n f, g\rangle \to 0 \quad \text{as } n\to\infty \text{ (recall } \textstyle\int f \int g = 0\text{)}.$$

Denote by $\bar{g}_1 := \int\limits_X g_1 \, d\mu$.

$$\langle (T\times S)^n f, g\rangle = \langle (T\times S^n) f, \overbrace{(g_1-\bar{g}_1)}^{g_1',\ \text{so } \int_X g_1' \, d\mu = 0} g_2\rangle + \langle (T\times S)^n f, \bar{g}_1 g_2\rangle.$$

$$\left| \frac{1}{n+1}\sum_{i=0}^{n} \langle (T\times S)^i f, g_1' g_2\rangle \right| = \left| \frac{1}{n+1}\sum_{i=0}^{n} \int\limits_{X\times Y} T^i f_1 \cdot S^i f_2 \cdot g_1' \cdot g_2 \, d\mu\times\nu \right|$$

$$= \left| \frac{1}{n+1}\sum_{i=0}^{n} \int\limits_X (T^i f_1 \cdot g_1') \, d\mu \cdot \int\limits_Y S^i f_2 \cdot g_2 \right|$$

$$= \left| \frac{1}{n+1}\sum_{i=0}^{n} \langle T^i f_1, g_1'\rangle \langle S^i f_2, g_2\rangle \right|$$

(Cauchy–Schwarz) $\quad \leq \|f_2\|_2 \|g_2\|_2 \cdot \frac{1}{n+1}\sum_{i=0}^{n} |\langle T^i f_1, g_1'\rangle|$

$(n\to\infty) \quad \to \|f_2\|_2 \cdot \|g_2\|_2 \cdot \int\limits_X f_1 \, d\mu \underbrace{\int\limits_X g_1' \, d\mu}_{=0}.$